INTRODUCTION TO
FINANCIAL AND ACTUARIAL
MATHEMATICS

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Chapter 1

BONDS

Bond or debenture is a debt instrument that obligates the issuer to pay to the bondholder the principal (the original amount of the loan) plus interest and serves as a device for indemnifying or guaranteeing an individual against loss.

Par / Face value is the initial value of a bond.

Maturity date - Bond maturity tells when investors should expect to get the principal back and how long they can expect to receive interest payments. The maturity of a bond can be any length of time, although typical bond maturities range from one year to 30 years. There are three groups of bond maturities: Short-term (bills) with maturities less than 1 year, medium-term (notes) with maturities between 1 and 10 years, and long-term bonds (bonds) with maturities greater than 10 years. Most bonds are 30 years or less, but bonds have been issued with maturities of up to 100 years. Some bonds never mature (perpetuities).

Yield to maturity (YTM) is the internal rate of return of a bond, if the security were to be held until maturity. It is a measurement of the performance of the bond. This technique in theory allows investors to calculate the fair value of different financial instruments. The calculation of YTM is identical to the calculation of internal rate of return. If a bond’s current yield is less than its YTM, then the bond is selling at a discount. If a bond’s current yield is more than its YTM, then the bond is selling at a premium. If a bond’s current yield is equal to its YTM, then the bond is selling at par.

Bond valuation is the process of determining the fair price of a bond. As
with any security, the fair value of a bond is the present value of the stream of cash flows it is expected to generate. Hence, the price or value of a bond is determined by discounting the bond’s expected cash flows to the present using the appropriate discount rate.

1.1 Zero–coupon Bonds

Zero–coupon bond is a bond, which does not bear interest, as such, but is sold at discounted price which is a substantial discount from the par value. The bondholder receives the full face value at maturity, and the ”spread” between the issue price and redemption price is the bond’s yield.

Formulae:

\[ P = \frac{F}{(1 + i)^n} \]  \hspace{1cm} (1.1)

\[ i = \sqrt[n]{\frac{F}{P}} - 1 \]  \hspace{1cm} (1.2)

where

- \( F \) \ldots face value / par value of a bond
- \( P \) \ldots purchase / fair price
- \( i (= YTM) \) \ldots yield rate (yield to maturity)
- \( n \) \ldots number of coupon periods to maturity

1.2 Coupon Bonds

Coupon bond is a bond issued with detachable coupons that must be presented to the issuer for interest payments.

Coupon rate is the nominal annual rate of interest paid on the face value of a bond.
Price interpolation - is a method used to determine the fair price of a coupon bond on a date which is between the dates when the coupons are paid out.

Aliquot interest is the interest that is due on a bond or other fixed income security since the last interest payment was made. This often occurs for bonds purchased on the secondary market, since bonds usually pay interest every six months, but the interest is accrued by the bondholders on a day-to-day basis. When a bond is sold, the buyer pays the seller the market price plus the accrued interest.

Formulae:

\[ P = F \cdot \left[ c \cdot a_{n|i} + \frac{1}{(1 + i)^n} \right] \]  
\[ I_a = \frac{c \cdot t \cdot FV}{360 \cdot 100} \]  
\[ C = F \cdot c \]  
\[ r = \frac{P}{F} \]  

where

\( F \) ... face value / par value of a bond  
\( P \) ... purchase value  
\( C \) ... coupon payment  
\( c \) ... coupon rate  
\( r \) ... bond rate  
\( i \) (\( = YTM \)) ... yield rate (yield to maturity)  
\( I_a \) ... accrued interest  
\( t \) ... time of accrued interest  
\( n \) ... number of coupon periods to maturity  
\( a_{k|i} \) ... Series Present Worth Factor (ordinary model)
Example 22: What was the purchase price of a coupon bond with a face value of CZK 10,000 maturing on 1 July 1998, having fixed coupon payments of 9 % p.a. payable on 1 July, if the bond was bought on 1 April 1995 with a yield rate of 12 %?

Solution:

price regarding 01/07/1994:

\[ F = \text{CZK } 10,000, \quad c = 0.09, \quad i = 0.12, \quad n = 4, \quad P = ? \]

\[
P_1 = 10,000 \cdot \left[ 0.09 \cdot a_4^{0.12} + \frac{1}{(1 + 0.12)^4} \right] = 10,000 \cdot \left[ 0.09 \cdot \frac{1 - (1 + 0.12)^{-4}}{0.12} + 0.7312 \right] =
\]

\[
= 10,000 \cdot \left[ 0.09 \cdot 3.03735 + 0.635181 \right] = 10,000 \cdot 0.908696 = 9,089
\]

price regarding 01/07/1995:

\[ F = \text{CZK } 10,000, \quad c = 0.09, \quad i = 0.12, \quad n = 3, \quad P_2 = ? \]

\[
P_2 = 10,000 \cdot \left[ 0.09 \cdot a_3^{0.12} + \frac{1}{(1 + 0.12)^3} \right] = 10,000 \cdot \left[ 0.09 \cdot \frac{1 - (1 + 0.12)^{-3}}{0.12} + 0.7312 \right] =
\]

\[
= 9,279
\]

price interpolation:

\[
P_{\text{int}} = P_1 + \frac{3}{4} \cdot (P_2 - P_1) = 9,089 + \frac{3}{4} \cdot (9,279 - 9,089) = 9,232
\]

accrued interest:

\[
I_a = \frac{3}{4} \cdot 10,000 \cdot 0.09 = 675
\]

final price:

\[
P = P_{\text{int}} + I_a = 9,232 + 675 = 9,907
\]

Answer:

The purchase price of the bond on 1st April 1995 was CZK 9,907.
Example 23: Find the bond rate of a bond on 1 October 1994, maturing 1 April 2002, with a face value of CZK 10,000, having fixed coupon payments of 12% p.a. paid out semi-annually on 1 April and 1 October, considering a yield rate of 10%.

Solution:

\[ F = \text{CZK} \ 10,000, \quad c = \frac{0.12}{2} = 0.06, \quad i = \frac{0.10}{2} = 0.05, \quad n = 15 \]

\[ P = 10,000 \cdot \left[ 0.06 \cdot a_{15|0.05} + \frac{1}{(1 + 0.05)^{15}} \right] = 11,038 \Rightarrow \frac{11.038}{10,000} = 1.1038 \]

Answer:
The bond rate of the given bond is 110.38%.

1.3 Callable Bonds

Callable bonds are bonds that may have a "call" provision that allows the issuer to pay back the debt (redeem the bond) before its nominal maturity date. When there is no such provision requiring a holder to let the issuer redeem a bond before its maturity date, the issuer may offer to redeem a bond early, and its holder may accept or reject that offer. European callable bonds can only be called on one specific call date, while American ones can be called at any time until the option matures.

Example 24: A $5,000 callable bond pays semi-annual interest at 10% p.a. and matures at par in 20 years. It may be called after 10 for $5,200. Which of the two possibilities give the highest fair price considering the interest yield of 8%?

Solution:

a) maturity in 10 years:

\[ F = \text{CZK} \ 5,000, \quad c = \frac{0.10}{2} = 0.05, \quad i = \frac{0.08}{2} = 0.04, \quad n = 20, \quad P = ? \]

\[ P = F \cdot \left[ c \cdot a_{n|i} + \frac{1}{(1+i)^n} \right] = 5,000 \cdot \left[ 0.05 \cdot a_{20|0.04} + \frac{1}{(1+0.04)^{20}} \right] = 5,680 \]
b) maturity in 20 years:

\[ F_1 = \text{CZK } 5,000, \quad F_2 = \text{CZK } 5,200, \quad c = \frac{0.10}{2} = 0.05, \quad i = \frac{0.08}{2} = 0.04, \quad n = 40, \quad P = ? \]

\[ P = F_1 \cdot c \cdot a_{n|i} + F_2 \cdot \frac{1}{(1+i)^n} = 5,000 \cdot 0.05 \cdot a_{40|0.04} + 5,200 \cdot \frac{1}{(1+0.04)^{40}} = 5,990 \]

**Answer:**

The highest fair price is of the bond with maturity in 20 years.

### 1.4 Exercises

**Ex.8.1:** What is the YTM of a zero-coupon bond with the face value of CZK 10,000 maturing in three years if it has been sold for CZK 7,512?

\[ [i=10 \%] \]

**Ex.8.2:** What is the purchase price of a zero-coupon bond with the face value of CZK 10,000 maturing in five years if it is required to have YTM of 15 %?

\[ [\text{CZK } 4,761] \]

**Ex.8.3:** What is the bond rate of a coupon bond with its nominal value of CZK 10,000 on the 15th April 2005, if the the bond matures on the 15th October 2010 and the coupons with the coupon rate of 10 % p.a. are paid semi-annually always on the 15th April and the 15th October? Consider an interest rate of reconcilable securities to be 4 % p.a.

\[ [r=1.29] \]

**Ex.8.4:** A $ 1,000 bond, redeemable at par on the 1st December 1998, pays semi-annual coupons at 9 %. The bond is bought on 1st July 1996. What is the purchase price if the desired yield is 8 % compounded semi-annually?

\[ [$ 1,022.26] \]

**Ex.8.5:** Find the fair price of a bond on the 16th October 1996, maturing on the 1st October 1998, with a face value of $ 2,000, having fixed coupon payments of 10 % p.a. paid out semi-annually on 1st April and 1st October, considering a yield rate of 9 %.

\[ [$ 2,082] \]
Ex.8.6: A bond with a par value of $100,000 has semi-annual coupons at the rate of 13%. It is purchased for a price of $94,703. At this price the purchaser who holds the bond to maturity will realize a semi-annual yield rate of 14%. In how many years will the bond mature? [10 years]
Chapter 2

SHARES AND OPTIONS

2.1 Share

Companies issue shares as a means of raising equity finance and determining ownership. Shares in public limited companies are traded on the Stock Market and as such the value of the shares will fluctuate depending on the demand.

Dividend is the distribution of profits to a company’s shareholders.

Shareholder / stockholder is an individual or company (including a corporation), that legally owns one or more shares of stock in a joint stock company and is therefore entitled to a share in its profits (dividends).

The price of shares quoted ex dividend does not include the right to receive next dividends.

Pre-emptive right is the right of current shareholders to maintain their fractional ownership of a company by buying a proportional number of shares of any future issue of common stock.

Shares of stock that are trading but no longer have pre-emptive rights to purchase additional shares at a price below the current market price are said to go or to be ex rights. Ex-rights shares are worth less than shares which are not yet ex-rights.
Formulae:

\[ PV = \frac{D_1}{(1+i)} + \frac{D_2}{(1+i)^2} + \cdots \quad (2.1) \]

Constant dividends \( D \):

\[ PV = \frac{D}{(1+i)} + \frac{D}{(1+i)^2} + \cdots = \frac{D}{i} \quad (2.2) \]

Annual growth rate of dividends \( g \) where \( g < i \):

\[ PV = \frac{D \cdot (1+g)}{(1+i)} + \frac{D \cdot (1+g)^2}{(1+i)^2} + \cdots = D \cdot \frac{1+g}{i-g} \quad (2.3) \]

Price of pre-emptive right before ex-rights date:

\[ R = \frac{PV_{\text{before}} - S}{N + 1} \quad (2.4) \]

Price of pre-emptive right after ex-rights date:

\[ R = \frac{PV_{\text{after}} - S}{N} \quad (2.5) \]

Expected rate of return:

\[ r_e = \sum_{i=1}^{n} r_i \cdot p_i \quad (2.6) \]

Risk:

\[ \sigma = \sqrt{\sum_{i=1}^{n} (r_e - r_i)^2 \cdot p_i} \quad (2.7) \]
where

\[ PV \ldots \text{purchase / fair price of a share} \]
\[ PV_{before} \ldots \text{price of shares before date shares go ex rights} \]
\[ PV_{after} \ldots \text{price of shares after date shares go ex rights} \]
\[ D \ldots \text{dividend} \]
\[ i \ldots \text{assumed interest rate} \]
\[ S \ldots \text{price of new shares without pre-emptive right} \]
\[ N \ldots \text{ratio of old shares to new shares} \]
\[ R \ldots \text{price of preemptive right} \]
\[ \sigma \ldots \text{risk of a share} \]
\[ r_e \ldots \text{expected rate of return of a share} \]
\[ r_i \ldots \text{particular rates of return} \]
\[ p_i \ldots \text{probability of particular rate of returns} \]

**Example 25:** What is the fair price of a share with dividends of CZK 100 and an interest rate of 8% p.a.

**Solution:**

\[ PV = \frac{D}{i} = \frac{100}{0.08} = 1,250 \]

**Answer:**

The fair price of the share is CZK 1,250.

**Example 26:** A corporation has already issued 20,000 shares and has decided to increase its capital by issuing 10,000 new shares of CZK 40 mil. What is the price of one pre-emptive right before the date the shares go ex rights, if the purchase price of the shares is CZK 4,600 and after the ex-rights date when the share price is CZK 4,200?

**Solution:**

\[ PV_{before} = \text{CZK 4,600}, \quad PV_{after} = 4,200, \quad R = ? \]

\[ N = \frac{\text{number of old shares}}{\text{number of new shares}} = \frac{20,000}{10,000} = 2 \]
\[ S = \frac{\text{capital increase}}{\text{number of new shares}} = \frac{40,000,000}{10,000} = 4,000 \]

before ex-rights date: \[ R = \frac{4,600 - 4,000}{2+1} = \frac{600}{3} = 200, \]

after ex-rights date: \[ R = \frac{4,200 - 4,000}{2} = \frac{200}{2} = 100 \]

**Answer:**

The price of one pre-emptive right before the date the shares go ex is CZK 200 and after the date the price is CZK 100.

**Example 27:** The following table shows the expected rate of return of two shares and the corresponding probability. Which of the shares is better to buy considering the total rate of return and the possible risk?

<table>
<thead>
<tr>
<th>rate of return a</th>
<th>rate of return B</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 %</td>
<td>5 %</td>
<td>20 %</td>
</tr>
<tr>
<td>7 %</td>
<td>8 %</td>
<td>30 %</td>
</tr>
<tr>
<td>13 %</td>
<td>8 %</td>
<td>30 %</td>
</tr>
<tr>
<td>15 %</td>
<td>6 %</td>
<td>20 %</td>
</tr>
</tbody>
</table>

**Solution:**

share A:
\[ r_e = 5 \cdot 0.20 + 7 \cdot 0.30 + 13 \cdot 0.30 + 15 \cdot 0.20 = 10 \]
\[ \sigma = \sqrt{(10-5)^2 \cdot 0.2 + (10-7)^2 \cdot 0.3 + (10-13)^2 \cdot 0.3 + (10-15)^2 \cdot 0.2} = 3.9 \]

share B:
\[ r_e = 20 \cdot 0.20 + 8 \cdot 0.30 + 8 \cdot 0.30 + 6 \cdot 0.2 = 10 \]
\[ \sigma = \sqrt{(10-20)^2 \cdot 0.2 + (10-8)^2 \cdot 0.3 + (10-8)^2 \cdot 0.3 + (10-6)^2 \cdot 0.2} = 5.65 \]

**Answer:**

It is better to buy share B as the risk is lower than in case of share A.
2.2 Option

Option is an intangible investment that gives the buyer the right to buy or to sell shares of a specified stock at a specified price on or before a given date.

Exercise / Strike price, is a key variable in a derivatives contract between two parties. Where the contract requires delivery of the underlying instrument, the trade will be at the strike price, regardless of the spot price (market price) of the underlying at that time.

In the case of a call option on a stock, the stock is the underlying / underlier and the option is the derivative. In the case of an option on a bond, the bond is the underlying.

Call option is a financial contract between two parties – the buyer and the seller of this type of option. The buyer of the option has the right but not the obligation to buy an agreed quantity of a particular commodity or financial instrument (the underlying instrument) from the seller of the option at a certain time for a certain price (the exercise price). The seller is obligated to sell the commodity or financial instrument if the option holder exercise the call option. The buyer pays a fee (called a premium) for this right.

Put option is a financial contract between two parties – the buyer and the seller of the option. The buyer of the option has the right but not the obligation to sell a commodity or financial instrument (the underlying instrument) to the seller of the option at a certain time for a certain price (the exercise price). The seller has the obligation to purchase at that exercise price, if the buyer does choose to exercise the option. The buyer pays a fee (called a premium) for this right.

Option underwriter is someone who guarantees the purchase of a full issue of an option.

Long position is the state of actually owning an option or another security, contract, or commodity.

Short position is the promise to sell a certain quantity of share or another security at a particular price in the future.
Formulae:

Profit of a call option holder (= loss of a call option subscriber):

\[ Z = \max(0, S - X) - c \quad (2.8) \]

Profit of a put option holder (= loss of a put option subscriber):

\[ Z = \max(0, X - S) - p \quad (2.9) \]

where

\[ Z \ldots \text{profit / loss} \]
\[ S \ldots \text{spot price} \]
\[ X \ldots \text{exercise price} \]
\[ c \ldots \text{price of call option} \]
\[ p \ldots \text{price of put option} \]

Example 28: Mr. Littler possesses a call option of a share underlier with an exercise price of CZK 3,000 and he paid CZK 100 for it. What is the loss or profit if the spot price of the share when the option is due is a) CZK 2,600; b) CZK 3,300?

Solution:

a) \( S = 2,600, \quad X = 3,300, \quad c = 100, \quad Z =? \)
\[ Z = \max(0, S - X) - c = \max(0; 2,600 - 3,000) - 100 = 0 - 100 = -100 \]

b) \( S = 3,300, \quad X = 3,000, \quad c = 100, \quad Z =? \)
\[ Z = \max(0, S - X) - c = \max(0; 3,300 - 3,000) - 100 = 300 - 100 = 200 \]

Answer:

In the first case, Mr. Littler lost CZK 100, in the other one he made a profit of CZK 200.
2.3 Exercises

**Ex.9.1:** What is the fair price of a share with dividends of the present value of CZK 100 and an expected annual increase of 2% taking into consideration an interest of 5%?

\[ \text{[CZK 3.400]} \]

**Ex.9.2:** The dividends of a share have been increasing by CZK 20 steadily for 5 years and have reached the value of CZK 200. What is the discounted value of one such a dividend if it is discounted five years back considering an interest rate of 4%?

\[ \text{[CZK 705]} \]

**Ex.9.3:** A corporation has already issued 30 thousand shares and has decided to increase its capital by issuing 10 thousand new shares of CZK 50 mil. What is the price of one pre-emptive right before ex-rights date, if the purchase price of the shares is CZK 5,600 and after the ex-rights date when the share price is CZK 5,200?

\[ \text{[CZK 150 (before); CZK 100 (after)]} \]

**Ex.9.4:** The given table shows the expected rate of return of two shares and the corresponding probability. Which of the shares is better to buy concerning the total rate of return and the possible risk?

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<td>10 %</td>
</tr>
<tr>
<td>5 %</td>
<td>5 %</td>
<td>30 %</td>
</tr>
<tr>
<td>6 %</td>
<td>4 %</td>
<td>40 %</td>
</tr>
<tr>
<td>3 %</td>
<td>2 %</td>
<td>20 %</td>
</tr>
</tbody>
</table>

\[ \text{[A is more profitable, B is safer]} \]

**Ex.9.5:** A call option holder owns an option of CZK 2,000 share underlier costing him CZK 80. What is the profit or loss if the price of the share is CZK 2,120?

\[ \text{[profit of CZK 40]} \]
**Ex.9.6:** Mr. Novotný bought one call and one put option of the same share underlier with an exercise price of CZK 2,000. For each of them he paid CZK 100. What is his profit if the market price of the share is CZK 2,600?  
[CZK 400]

**Ex.9.7:** Mr. Novák bought one call option for CZK 80 and one put option of the same share underlier with an exercise price of CZK 1,000. What was the price of the put option if his profit is CZK 200 and the spot price of the share is CZK 1,300?  
[CZK 20]